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LETTER TO THE EDITOR

On a certain lower bound to the effective potential of $\lambda\phi^4$

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Abstract. The lower bound to the effective potential of the $\lambda\phi^4$ field theory obtained by Caswell and Tarrach is shown to be identical to the saddle-point approximation of the path integral in the auxiliary field representation. The accuracy of this approximation is shown to be low.

1. Introduction

In two recent publications [3, 5], an analytical lower bound to the effective potential of $\lambda\phi^4$ in d spacetime dimensions was obtained. The reasoning is very simple and is quickly reconstructed here: if λ is positive, then we have for every ϕ and every positive M the following inequality

$$\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \geqslant \frac{M^2}{2}\phi^2 - \frac{3}{2}\frac{(M^2-m^2)^2}{\lambda} \tag{1}$$

which has a simple geometrical explanation [3]. In a short notation, we can write

$$V(\phi) \geqslant \underline{V}(\phi, M). \tag{2}$$

If we now take an arbitrary wavefunction $|\phi\rangle$, normalized to unity and centred around the constant field φ, we then have the inequality

$$\langle\phi|\hat{H}|\phi\rangle > \langle\phi|\underline{\hat{H}}|\phi\rangle \tag{3}$$

with $\hat{H}$ and $\underline{\hat{H}}$ the Hamiltonians corresponding to V and $\underline{V}$, respectively. Both expectation values in (3) lie above their respective minimum, and it is easy to see that

$$\min_{\psi}\langle\psi|\hat{H}|\psi\rangle > \min_{\psi}\langle\psi|\underline{\hat{H}}|\psi\rangle \tag{4}$$

where we minimize over all ψ's subject to the above-mentioned constraints. The right-hand side is simply the ground-state energy of a harmonic oscillator with mass M, shifted by a constant. If we extract the volume of space, we get the quantity

$$E(\varphi, M) = I_1(M) + \frac{M^2}{2}\varphi^2 - \frac{3}{2}\frac{(M^2-m^2)}{\lambda} \tag{5}$$

with

$$I_1(M) = \frac{1}{2}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{d-1}}\sqrt{\boldsymbol{p}^2+M^2}. \tag{6}$$

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The left-hand side of (4), per unit space-volume, is the standard definition of the effective potential in field theory, and (4) then becomes

$$V_{\text{eff}}(\varphi) > E(\varphi, M). \tag{7}$$

This is valid for all positive M, so we take the maximum to find

$$V_{\text{eff}}(\varphi) > \underline{V}_{\text{eff}}(\varphi) \tag{8}$$

with

$$V_{\text{eff}}(\varphi) = \max_{M} E(\varphi, M). \tag{9}$$

Using (5) and (6), this yields the following equation for the best value of M:

$$\frac{\lambda}{6}[I_0(M) + \varphi^2] = M^2 - m^2 \tag{10}$$

with

$$I_0(M) = \frac{1}{2}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{d-1}} \frac{1}{\sqrt{\boldsymbol{p}^2 + M^2}}. \tag{11}$$

2. Alternative derivation

We can now link these results to the well known formulation of $\lambda\phi^4$ using an auxiliary field [2]. We start from the generating functional [8]:

$$Z[J] = \int \mathrm{D}\phi \exp\left[-S(\phi) + \int J \cdot \phi \right] \tag{12}$$

with the Euclidean action in d dimensions given by

$$S(\phi) = \frac{1}{2}\int \phi(-\Box + m^2)\phi + \frac{\lambda}{4!}\int \phi^4. \tag{13}$$

The resulting quartic term in the integrand in (12) can be written as follows

$$\exp\left[-\frac{\lambda}{4!}\int \phi^4 \right] = \int \mathrm{D}\sigma \exp\left[-\frac{1}{2}\int \sigma^2 - \frac{\alpha}{2}\int \sigma \cdot \phi^2 \right] \tag{14}$$

where

$$\alpha^2 = -\frac{\lambda}{3}. \tag{15}$$

This yields the path integral

$$Z[J] = \int \mathrm{D}\sigma \mathrm{D}\phi \exp\left[-\frac{1}{2}\int \phi(-\Box + m^2 + \alpha\sigma)\phi - \frac{1}{2}\int \sigma^2 + \int J \cdot \phi \right]. \tag{16}$$

We can then perform the saddle-point approximation to obtain the one-loop approximation to the effective action. This is a straightforward textbook exercise [8] and we find

$$\Gamma^{1\ell}(\phi) = \frac{1}{2}\int \phi(-\Box + m^2 + \alpha\sigma)\phi + \frac{1}{2}\int \sigma^2 + \frac{1}{2}\mathrm{tr}\ln(-\Box + m^2 + \alpha\sigma) \tag{17}$$

with σ such that (17) is stationary with respect to σ. The effective potential is the value of the effective action for a constant field, per unit spacetime volume. We thus have

$$V_{\text{eff}}^{1\ell}(\varphi) = \frac{1}{2}(m^2 + \alpha\sigma)\varphi^2 + \frac{\sigma^2}{2} + \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \ln(p^2 + m^2 + \alpha\sigma). \tag{18}$$

We can rewrite this in terms of the new parameter

$$M^2 \equiv m^2 + \alpha\sigma \tag{19}$$

which gives us

$$V_{\text{eff}}^{1\ell}(\varphi) = \frac{M^2}{2}\varphi^2 + \frac{1}{2}\left[\frac{M^2 - m^2}{\alpha}\right]^2 + \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \ln(p^2 + M^2). \tag{20}$$

This still has to be minimized with respect to M^2 (instead of σ). This yields the condition

$$\frac{1}{2}\varphi^2 + \frac{M^2 - m^2}{\alpha^2} + \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^2 + M^2} = 0 \tag{21}$$

or, using (15),

$$M^2 - m^2 = \frac{\lambda}{6}\varphi^2 + \frac{\lambda}{6}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^2 + M^2}. \tag{22}$$

Since $p^2 = p_0^2 + \boldsymbol{p}^2$, with $\boldsymbol{p}$ a momentum vector in $d-1$ space dimensions, we can integrate over p_0 in (22) to obtain

$$M^2 - m^2 = \frac{\lambda}{6}\varphi^2 + \frac{\lambda}{6}\left[\frac{1}{2}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{d-1}} \frac{1}{\sqrt{\boldsymbol{p}^2 + M^2}}\right] \tag{23}$$

which is identical with equation (10). Furthermore, as regards the effective potential itself, we can integrate over p_0 in the last term in (20) and use (15) again on the second term to obtain

$$V_{\text{eff}}^{1\ell}(\varphi) = \frac{M^2}{2}\varphi^2 - \frac{3}{2}\frac{(M^2 - m^2)^2}{\lambda} + \frac{1}{2}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{d-1}} \sqrt{\boldsymbol{p}^2 + M^2} \tag{24}$$

which agrees with $V_{\text{eff}}(\varphi)$ in (5). This shows the equivalence of both methods. The advantage of the path integral method is that it can be easily extended to higher-order calculations while the approach in [3] and [5] cannot. The link between the inequality (1) and the saddle-point approximation (16) and (17) can be directly made through the use of path integrals. We can use the parametrization (19) to rewrite (1) as

$$\exp -V(\phi) \leqslant \exp -\left[m^2\phi^2 + \frac{\alpha\sigma_0\phi^2}{2} + \frac{\sigma_0^2}{2}\right] \tag{25}$$

valid for all purely imaginary σ_0. This implies directly that

$$Z[J] \leqslant \int \mathrm{D}\phi \exp\left[-\frac{1}{2}\int \phi(-\Box + m^2 + \alpha\sigma_0)\phi - \frac{1}{2}\int \sigma_0^2 + \int J\cdot\phi\right]. \tag{26}$$

It is easy to show that (24) is not even correct to first order in λ for small coupling. A bit of algebra shows that (22) and (20) generate, for $\varphi = 0$, the following term of order λ in $V_{\text{eff}}^{1\ell}$,

$$\frac{\lambda}{24}[I_0(m)]^2 \tag{27}$$

while the correct contribution to V_{eff} should be three times as large, as seen from applying perturbation theory. The strong coupling regime, $\lambda \to \infty$, is not well described either. In the $(0+1)$-dimensional case (quantum mechanics), all integrals in (20) and (22) are finite, and a simple calculation produces the following scaling behaviour for $V_{\text{eff}}^{1\ell}(0)$ (which is an estimate to the ground-state energy):

$$E_0^{1\ell} \to 0.472\,470\left(\frac{\lambda}{4!}\right)^{1/3} \qquad \lambda \to \infty \tag{28}$$

which is 30% lower than the well known result [1] obtained through high-precision methods. The deviations on these two coefficients (for small λ and for big λ) are of course a direct consequence of the lower bound property.

3. Conclusion

We have shown that the lower bound to the effective potential of $\lambda\phi^4$ field theory coincides with the saddle-point approximation in the σ-field representation. It is straightforward to calculate corrections to this lower bound systematically, although it is no longer guaranteed that it remains a lower bound. On the other hand, the Gaussian effective potential provides an upper bound and here also, there are ways of systematically calculating corrections using either optimized perturbation theory [4] or a variational method for $\lambda\phi^4$ based on an effective action for local composite operators [6, 7]. An intriguing possibility would be that combining both approaches in some way might provide a fastly converging non-perturbative method in quantum field theory.

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